Reliability Approaches in the Design of Slopes

by

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Abstract

Reliability theory is a powerful tool for investigating cases in which the values of parameters are not known exactly or where the engineer wishes to know the uncertainty associated with the calculations. Because it is seldom possible to compute the uncertainty in the factor of safety directly, several approximate techniques have been developed, and these are illustrated by a simple example. A major task in performing a reliability analysis is determining the uncertainty in the relevant parameters. It is important to distinguish between local uncertainties and systematic uncertainties and to incorporate averaging effects. The theory can also be extended to estimate the contribution of seismic hazards to the overall risk of slope failure.

Introduction

Using relatively crude strength data obtained from a preliminary boring program, an engineer calculates that the factor of safety of a slope is 1.55. The engineer then carries out a program of further investigation and testing to improve the estimates of the soil properties and finds that the computed factor of safety becomes 1.40. The engineer is much more confident of the latter result because it is based on better data, but how is he or she to convince the regulatory authorities to be satisfied with the smaller value?

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A mining company operates a large tailings pond. One day the maintenance crew notices some possible signs of distress a one corner of the embankment that retains the tailings. The consequences of a possible failure include serious damage to the environment, but stopping the filling of the pond requires closing production and major losses in revenue. The management would like to make a decision based on probabilistic assessment of the alternatives. How is the geotechnical engineering staff to respond to their request for the probability of failure?

A plant is located near a slope in a Karst terrain. From historical records it is possible to estimate the annual rate of occurrence of landslides in the geologic formation, and the management of the plant is using that estimate in its decisions about how to expand the plant and whether to cut back the slope. The management asks the consulting engineer what effect the earthquake hazard will have on the probability of failure. How can the effect be estimated?

Each of the above cases is loosely based on an actual situation, and most engineers can add other examples to the list. Fortunately, reliability theory provides a well established way to attack such problems. As geotechnical engineers are increasingly called on to quntify their confidence or uncertainty, reliability theory has become more important in geotechnical practice. The following pages describe the basis of reliability approaches to geotechnical problems, show how it can be applied in practice, and discuss some of the current problems in its use.

Reliability Theory

As the name implies, reliability theory describes analytically how reliable a facility or structure is. Although there are several different ways of expressing the theory, most start by considering the situation illustrated in Figure 1. Based on the best estimates of the geometry and soil properties, an engineer has calculated the factors of safety against failure for two slopes. One has an estimate of 1.2, and the other has an estimate of 1.5. However, the engineer is much more confident of the result for the first slope because the soil properties and the geometry of the strata and failure surfaces are well known. The standard deviation of the factor of safety is estimated to be 0.1. In the other case the best estimate of the factor of safety is 1.5, but the standard deviation of the factor of safety is 0.5.

Figure 1 shows the probability density functions for the reasonable assumption that the factors of safety are normally distributed. The probability of failure in each case is the area under the probability density function to the left of the vertical line at F. S. = 1.0. It is clear that the area for the second case is much larger. In other words, the probability of failure is greater for the case with the larger factor of safety.

This result conforms to geotechnical experience. Geotechnical engineers know that the factor of safety by itself does not provide enough information to evaluate the safety of a slope and that the criteria for a satisfactory value depend on the degree of confidence that the engineer has in the information that went into the calculation. Reliability theory is one way to express that confidence.

The *reliability index* is represented by the Greek letter β and defined by the equation:

$$\beta = \frac{E[F] - 1.0}{\sigma_F} . \qquad (1)$$

In this equation, E[F] is the estimated value of the factor of safety (usually the value calculated from the best estimates of the parameters), and σ_F is the standard deviation of the factor of safety. In some cases β is defined in terms of the driving or loading forces and the resisting forces:

$$\beta = \frac{E[R] - E[Q]}{(\sigma_R^2 + \sigma_Q^2)^{1/2}}, \quad (2)$$

where R and Q stand for the resistance and the load, respectively. Properly applied, the two formulations are equivalent, but computational considerations may dictate using one or the other.

If the factor of safety is normally distributed, the probability of failure, that is the probability that $F \leq 1.0$, is easily computed as 1.0 minus the cumulative distribution function for a normal distribution with mean value of 0.0 and standard deviation of 1.0 evaluated at β. Figure 2 shows the probability of failure for a range of values of the reliability index. Similar plots can be created for other distributions; they are close to the values in Figure 2 for β less than about 2.

The reliability index provides a measure of how much confidence the engineer can have in the computed value of the factor of safety and leads to an estimate of the probability of failure. It should be borne in mind that this probability is actually the lower bound on the probability of failure because it ignores all the factors that have not been considered by the engineer. A great many failures result from the failure to consider a possible mode of behavior rather than from failure to use the correct parameters in design. With that proviso in mind, the question is then how to compute the reliability index.

There are several methods of calculating the value of the standard deviation of the factor of safety, σ_F, or, equivalently, its variance VAR_F , which is simply the square of σ_F. They include:

- Direct calculation. In some cases the equation for the factor of safety or for the margin of safety can be solved analytically to give the variance of the factor of safety. In particular, if the margin or factor of safety can be expressed as the sum of the random variables, the variance is simply the sum of the variances of each contributing variable. Unfortunately, such cases are rare.

- Monte Carlo simulation. Modern computers make it possible to generate a large number of solutions in which each of the variables is allowed to vary randomly according to its prescribed probability distribution. The technique is easy to program on a computer; in many cases it can be performed using a modern spreadsheet. Its main advantage is simplicity. Its major disadvantage is that it may be necessary to run a very large number of cases to get robust results.
- First Order Second Moment (FOSM) methods. These techniques are based on using the first terms of a Taylor series approximation for the variance. They give a direct approximation for σ_F and at the same time provide direct insight into the relative contributions of the various parameters. The basic method is easy to use. Its major disadvantage is that it gives an approximate solution that may be quite inaccurate.
- The Hasofer-Lind method. Hasofer and Lind (1974) developed an improvement to the FOSM method to account for the fact that the derivatives used in the FOSM should be evaluated at a point on the failure surface, that is, when F = 1.0. When the underlying assumptions of the FOSM method are satisfied, this technique improves the accuracy of the results. However, it does require an iterative procedure that is not intuitively obvious, and, when the FOSM assumptions are not reasonable, the technique can actually give results that are less accurate than other techniques. Ang and Tang (1990) give a

very clear description of the computational procedures.

- The point estimate method. Wolff (1996) has described this approximate technique, which has been widely adopted by the U. S. Army Corps of Engineers (e. g. U. S. Army 1995). It involves computing the factor of safety using values of each variable that are one standard deviation above or below the expected value and combining the results to estimate the expected value and standard deviation of the factor of safety. For N variables this requires 2^N computations, which is much less than the number needed for a Monte Carlo simulation. The method is simple but approximate. When the variables are not normally or log-normally distributed, it may not be intuitively obvious how to choose the paris of values to be used.

The best way to illustrate the use of reliability theory and the various methods of performing the calculations is to consider an example. This is done in the next section.

An Example Using the Culmann Method

The Culmann method is used to analyze the stability of a slope such as that illustrated in Figure 3. This method is limited by its assumption that the surface of sliding is a plane, so other techniques such as the various methods of slices have superseded it in engineering practice. Nevertheless, it remains valid for cases in which there are pre-existing planes of weakness such as joints. Since it combines some of the complexity of slope stability calculations with computational formulas that are simple enough to be manipulated, it serves to illustrate the application of reliability theory to the problem of slope stability.

With the notation defined in Figure 3, the factor of safety for a condition in which water pressures do not apply is found to be

$$F = \frac{c + \dfrac{1}{2}\dfrac{\gamma H}{\sin\psi}\sin(\psi - \theta)\cos\theta\tan\phi}{\dfrac{1}{2}\dfrac{\gamma H}{\sin\psi}\sin(\psi - \theta)\sin\theta} . \tag{3}$$

In this equation the notation, in addition to that defined in Figure 3, is that c and ϕ are the strength parameters and γ is the unit weight of the soil or rock. For purposes of illustration, let it be assumed that the inclination of the failure plane is known — perhaps because the joint set has been well established. This leaves three parameters whose values are uncertain: the two strength parameters and the unit weight.

Two cases will be used for illustration. It will be assumed that the variables are all normally distributed. Table I shows the means and the coefficients of variation for the uncertain parameters. (The coefficient of variation, COV, is the ratio between the standard deviation and the mean of a variable.) The COV for the friction component refers to the tangent of ϕ, not to ϕ itself. The COVs for the friction correspond approximately to COVs of 0.1 and 0.2 for cases 1 and 2, respectively. Substituting the mean values into equation [3] gives factors of safety of 1.29 and 1.2.14 for the two cases, respectively. However, it is clear that the values of the parameters are much less well determined in the second case. As the uncertain parameters appear in both the numerator and the denominator of equation [3], direct calculation if σ_F is not possible. The use of the other techniques is described in the following sections.

TABLE I

PARAMETERS FOR EXAMPLE CASES

Variable	Units	Case 1		Case 2	
		Mean	COV	Mean	COV
γ	KN/m^3	22	0.1	22	0.2
c	KN/m^2	5	0.1	5	0.4
ϕ	degrees	15		30	
tan ϕ		0.267949	0.07	0.577350	0.24

Monte Carlo Simulation

Monte Carlo simulation involves generating a large number of sets of values of the parameters using a random number generator modified so the resulting distributions agree with the desired distributions of the parameters. In this case all the parameters are normally distributed, and the means and standard deviations can be found from Table I. The simulation can be done on any modern spreadsheet. Two simulations with 200 sets of values each were done for Case 1. The first simulation gave E[F] = 1.295, σ_F = 0.1012, and β = 2.915. The second simulation gave E[F] = 1.309, σ_F = 0.0950, and β = 3.251. A single simulation for case 2 gave E[F] = 2.137, σ_F = 0.4985, and β = 2.280.

Although the number of sets of values is rather small, these results illustrate several points. First, they show that the reliability index for the first case is larger than for the second case, although the E[F] is much larger in the second case. In other words, the uncertainty in the parameters in the second case makes the slope significantly less reliable. The probabilities of failure corresponding to these values of β are 1.78×10^{-3} and 0.58×10^{-3} for the two simulations in Case 1 and 11.3×10^{-3} for Case 2. Case 2 has a probability of failure that is about one order of magnitude larger than Case 1.

Second, different simulations for Case 1 gave different answers, so the user must be aware of the uncertainty that is inherent in Monte Carlo simulation. This can be reduced by taking a larger set of samples in the simulation, but it remains a drawback of Monte Carlo simulation.

First Order Second Moment

The FOSM method estimates the standard deviation in F by expanding F in a Taylor series. The details can be found in most texts on reliability theory (e. g. Ang and Tang 1990). The result is, for the case that the variables are not correlated:

$$\sigma_F^2 \approx \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i} \right)^2 \sigma_{x_i}^2 . \tag{4}$$

The derivatives can be evaluated analytically or numerically. One of the advantages of this approach is the possibility of using numerical differences to approximate the derivative in complicated situations for which analytical differentiation may be very difficult. Another very important advantage is that, for each parameter, the method shows the combined contribution of the derivative and the variance and thus identifies clearly those variables that have the strongest influence on the uncertainty. Further studies should then concentrate on those variables that have the largest influence.

The FOSM method gives $\beta = 3.117$ for Case 1 and 2.513 for Case 2. the corresponding probabilities of failure are 0.91×10^{-3} and 5.99×10^{-3}, respectively. The relative reliability of the two cases remains the same, but the numbers are somewhat changed from the simulation results.

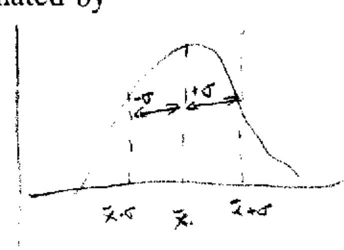
Hasofer-Lind

The Hasofer-Lind method is a refinement of the FOSM method. It starts from a recognition that the right hand side of equation [4] is actually the second term in the Taylor series and that the first term can be ignored only if the derivatives are evaluated at a point where $F = 1.0$. Finding the nearest such point involves an iteration that will not be described here. However, the results for the two cases are $\beta = 3.436$ and 2.574, respectively. The corresponding probabilities of failure are 0.30×10^{-3} and 5.03×10^{-3}.

Point Estimate Method

In the point estimate method the analyst selects two values for each variable. One is at the mean plus one standard deviation, and the other is at the mean minus one standard deviation. Since there are N variables and two points for each, there are 2^N combinations. In the present case $N = 3$, so there are eight combinations, for each of which the factor of safety is calculated. Each of the combinations is weighted equally. Therefore, the estimate of the mean is simply the sum of the results divided by the number of combinations, in this case 8. Similarly, the sum of the squares of each result is also divided by the number of combinations. If the result of each combination is designated F_i, the variance can be estimated by

$$\sigma_F^2 \approx \frac{1}{N}\sum_{i=1}^{N} x_i^2 - \left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)^2 . \qquad (5)$$



Equation [5] looks much more formidable than the actual calculations turn out to be.

The point estimate method yields $\beta = 3.149$ for Case 1 and 2.519 for Case 2. The corresponding probabilities of failure are 0.82×10^{-3} and 5.88×10^{-3}. Again the relative values of β and the probability are similar, but the numbers differ from those from the other methods.

Summary of Numerical Results

Table II summarizes the results for the two cases from the different methods of calculation.

The results show clearly that Case 2 is much more reliable than Case 1 by any method of calculation. However, the actual numerical results do differ between the various methods of calculation. In this instance the Hasofer-Lind method, which is intended to be an improvement on the FOSM method, seems to give the largest values of β and the smallest probabilities of failure.

Table II demonstrates that, if one is making comparisons between alternative designs, the comparisons should be made using the same methods of computation. Of course, the same observation could also be made if the comparison were to be made using factors of safety; the different techniques for calculating the factor of safety do not give the same or even comparable results. Another conclusion is that, however the reliability is evaluated, the slope in Case 1, which has a much smaller mean value of the friction angle, far more reliable and far less likely to fail than the slope in Case 2. Since the aim of calculation should be to establish the relative reliability of a structure, not simply an arbitrarily defined factor of safety, the results in Table II are more meaningful than a simple calculation of factor of safety.

TABLE II

SUMMARY OF NUMERICAL RESULTS FOR EXAMPLE PROBLEM

	E[F]	σ_F	β	p{[f]
Case 1				
FOSM	1.294	0.0942	3.117	0.91×10^{-3}
H - L	1.294	0.0725	3.436	0.30×10^{-3}
Point Est.	1.299	0.0950	3.149	0.82×10^{-3}
Monte Carlo	1.295	0.1012	2.915	1.78×10^{-3}
Case 2				
FOSM	2.144	0.4550	2.513	5.99×10^{-3}
H - L	2.144	0.4358	2.574	5.03×10^{-3}
Point Est.	2.167	0.4631	2.519	5.88×10^{-3}
Monte Carlo	2.137	0.4985	2.280	11.30×10^{-3}

<u>An Exact Case</u>

If the unit weight is completely known so that its variance is zero, then all the uncertain variables appeal in the numerator of equation [3]. The variance of the sum or difference of two normally distributed variables is simply the sum of the variances of each variable, so this case can be solved exactly. The result for Case 1 is that $\beta = 3.867$, and the probability of failure becomes 0.055×10^{-3}. The FOSM, Hasofer-Lind, and point estimate methods all give the same result. One trial of Monte Carlo simulation gives $\beta = 3.816$, but it is to be expected that the simulation results will show some scatter. For Case 2, when the unit weight is certain, $\beta = 2.592$, and the probability of failure is 4.77×10^{-3}. Again the FOSM, Hasofer-Lind, and point estimate methods give the same result, but Monte Carlo simulation is slightly in error with $\beta = 2.620$. These results how that, as one would expect, when the uncertainty in one parameter is reduced the overall uncertainty is also reduced.

Uncertainty in the Parameters

Once the mechanics of calculating the reliability index are understood, the next problem is how to estimate the uncertainties in the parameters that go into the calculation. In the example described above all the uncertain parameters were soil properties. In a more general and more realistic case there should also be uncertainty in the location or size of different elements of the soil profile, in the pore fluid pressure,or in other aspects of the analysis. These uncertainties can be treated in the same way that the uncertainties were treated in the example, except that numerical approximations become necessary as more uncertain parameters are introduced.

The uncertainty of any parameter can be divided into two components: *scatter* and *systematic error*. Scatter is the random variability is a parameter due to inherent spatial variability and errors in measuring the values of the parameter. It can be considered to involve the random variation about the mean. Systematic error is the uncertainty that applies across all locations. It is the uncertainty in the value of the mean itself.

Scatter is itself composed of two components: *real spatial variability* and *random testing* *errors* or *noise*. Real spatial variability is, as the name implies, something that is inherent in the nature of the soil or rock profile. Random testing error is an artifact of the testing process. Since only the real spatial variability contributes to the uncertainty of the actual facility, it is necessary to separate it from the noise and to eliminate the noise from the estimate of the uncertainty. Christian et al. (1994) describe one widely used method for doing this. It starts by finding how the values of the parameter are correlated over distance, a relation that is called the autocovariance function. This is then extrapolated back to a distance of zero; the result should be the local variance of the parameter. Usually this is found to be less than the measured value of the variance, and the difference must be due to measurement error because the measurement error must be purely local and uncorrelated over distance. When the measurement error is subtracted from the measured variance, the result is the variance due to real spatial variance. This procedure also gives a measure of the correlation between values of the parameter over distance, which is useful in the subsequent procedures.

Systematic error can also be divided into two parts: *statistical error in the mean* and *measurement bias*. Statistical error in the mean arises because even an exact measurement process that takes a finite number of samples will have some error in the estimate of the mean. A well known way to estimate this error is to divide the computed variance in the parameter by the number of points used to evaluate it. This gives a measure of how uncertain the mean value is. Measurement bias depends on the particular measurements used. It arises because many engineering parameters are evaluated by tests that measure something other than the actual parameter of interest. For example, the field vane test does not measure the shear strength on failure surfaces likely to be involved in a failure, but its results can be converted into meaningful numbers. The errors in that conversion make up the measurement bias.

Thus the engineer must identify, for each parameter, how the uncertainty is divided between spatial variation and systematic error. Sometimes this must depend on judgment. Sometimes the analysis dictates how the division must be done. For example, the thickness of a stiff crust may be uncertain, and the uncertainty would be divided into the two categories. On the other hand, the depth to a stiff lower stratum might also be uncertain, but it would be considered entirely a systematic error because it limits the location of the failure surface and this applies across the entire geometry of the analysis.

The main reason for separating the uncertainty into two categories is that the spatial variables tend to be averaged out over the failure surface, or over the geometry that applies to any other analysis. If L is the significant length of the analysis (for example, the length of the failure surface) and r_0 is the *autocorrelation distance* (the distance over which the spatial variability decreases to 1/e of its value at $r = 0$), the it has been found that the ratio between the average variance of the parameter to be used in analysis to the computed spatial variance is approximately $2r_0/L$. When the failure surface is large compared to the correlation distance, this has the effect of removing most of the spatial variability from the calculation.

Another factor that should be considered is the error in the analytical model. The model error has two effects. First, the model can be expected to be in error by some average amount or ratio, and this factor can be applied directly to the computed mean value of the estimated factor of safety. Second, the uncertainty in the model results, expressed as a variance, can be added to the computed variance of the factor of safety. These effects go in opposite directions if the analytical model is believed to be conservative. The effect on the mean is to raise the value of F, which increases β. The uncertainty increases the variance of F, which decreases β. Christian et al. (1994) show one case in which the reliability increases and two in which it decreases.

Application to Earthquake Hazard

An interesting extension of reliability theory arises when one wishes to estimate the effect of seismic hazard on the probability of failure. The details are described by Christian and Urzua (1997), but the basic procedure is summarized as follows. First, historical observation of failures leads to an estimate of the annual probability of failure. This can be converted into value of β, and a reasonably conservative value of σ_F leads to an estimate of the static F.

Equation [3] can be extended to include the effects of a horizontal acceleration factor. Algebraic manipulation of the new equation and equation [3] leads to

$$F^* = \frac{F - A a_h \tan\phi}{\frac{A a_h}{\tan\theta} + 1} \tag{6}$$

where F* is the dynamic factor of safety, F is the static factor of safety, A is the amplification factor, a_h is the horizontal acceleration factor, and the other parameters have been identified. If it is assumed that the variance in F* is the same as in F, the probability of failure (F* less than 1.0) can be expressed as a function of the acceleration. The probability distribution of the accelerations can be determined from seismic hazard analyses or maps.

The analysis consists then of combining equation [6] with the annual probability of accelerations at different levels. The result is an annual probability of failure due to the earthquake hazard alone. Results for one area of moderate seismicity are that the annual hazard of failure is increase by a factor of only about 0.1 to 0.15.

Conclusions

Reliability theory is a powerful tool. It allows the engineer to investigate cases in which the values of parameters are not known exactly or where the engineer wishes to know the uncertainty associated with the calculations. The results are in a form that can be combined with other estimates of risk and reliability. Because it is seldom possible to compute the uncertainty in the factor of safety directly, several approximate techniques have been developed, and these are illustrated by a simple example. A major task in performing a reliability analysis is determining the uncertainty in the relevant parameters. It is important to distinguish between local uncertainties and systematic uncertainties and to incorporate averaging effects. Local uncertainties are often largely averaged out, and the analytical methods need to account for this. The theory can also be extended to estimate the contribution of seismic hazards to the overall risk of slope failure.

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Scatter
- Real spatial variability
- Random testing errors or noise

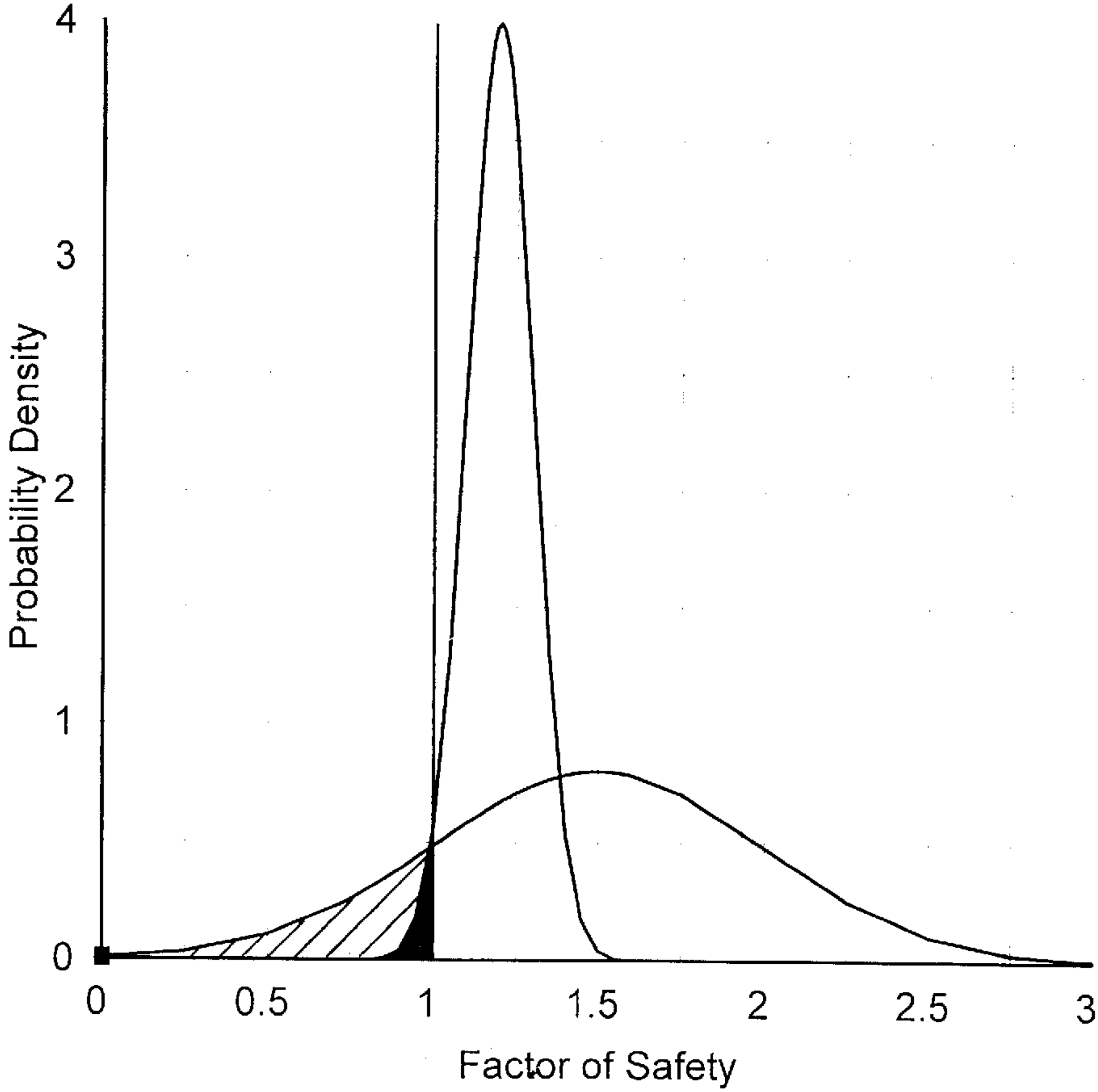

Figure 1. Probability density functions for two cases. The case with E[F] = 1.2 and $\sigma_F = 0.1$ has a lower total probability of failure than the case with E[F] = 1.5 and $\sigma_F = 0.5$.

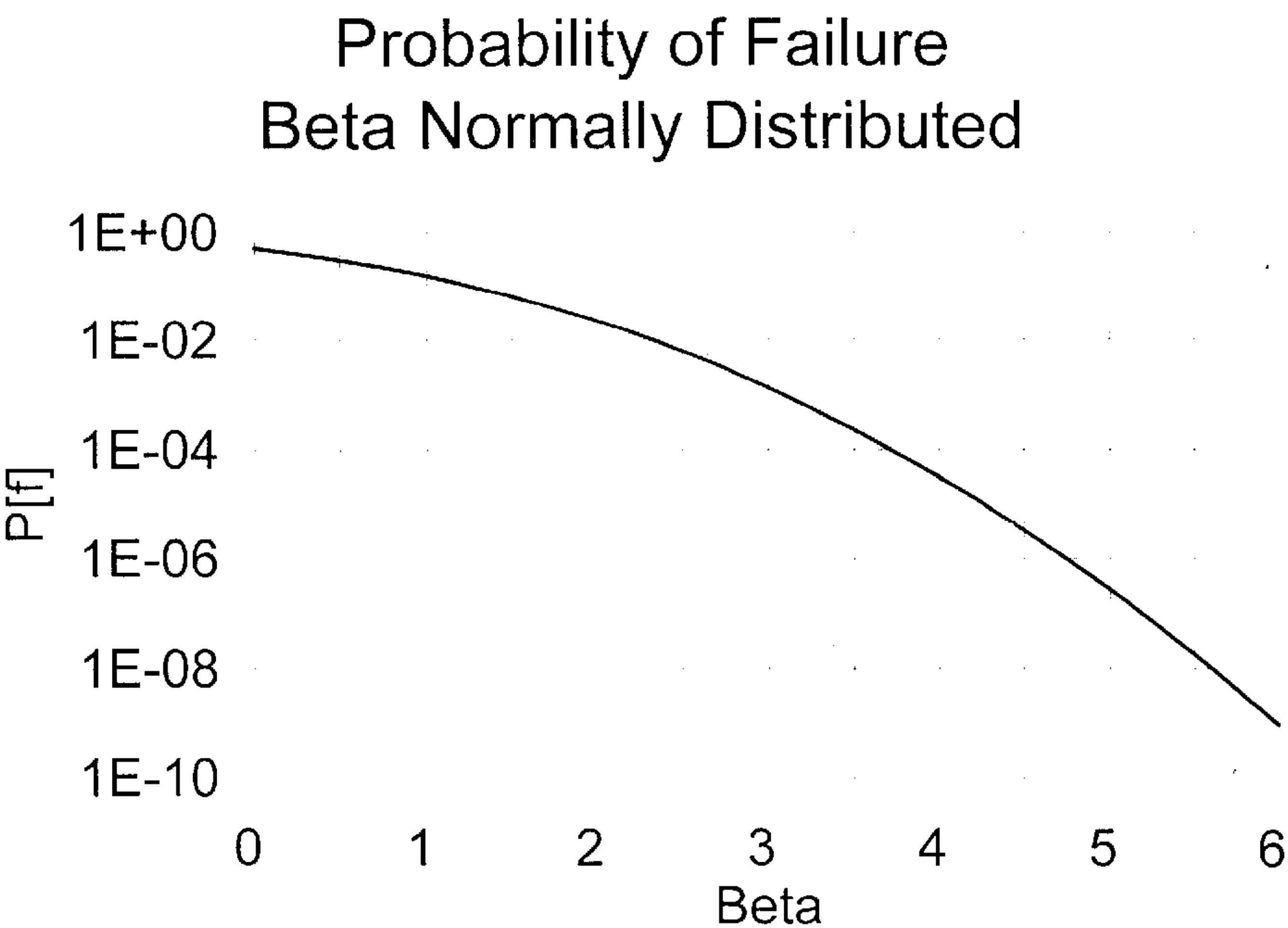

Figure 2. Probability of failure as a function of reliability index for normally distributed factor of safety.

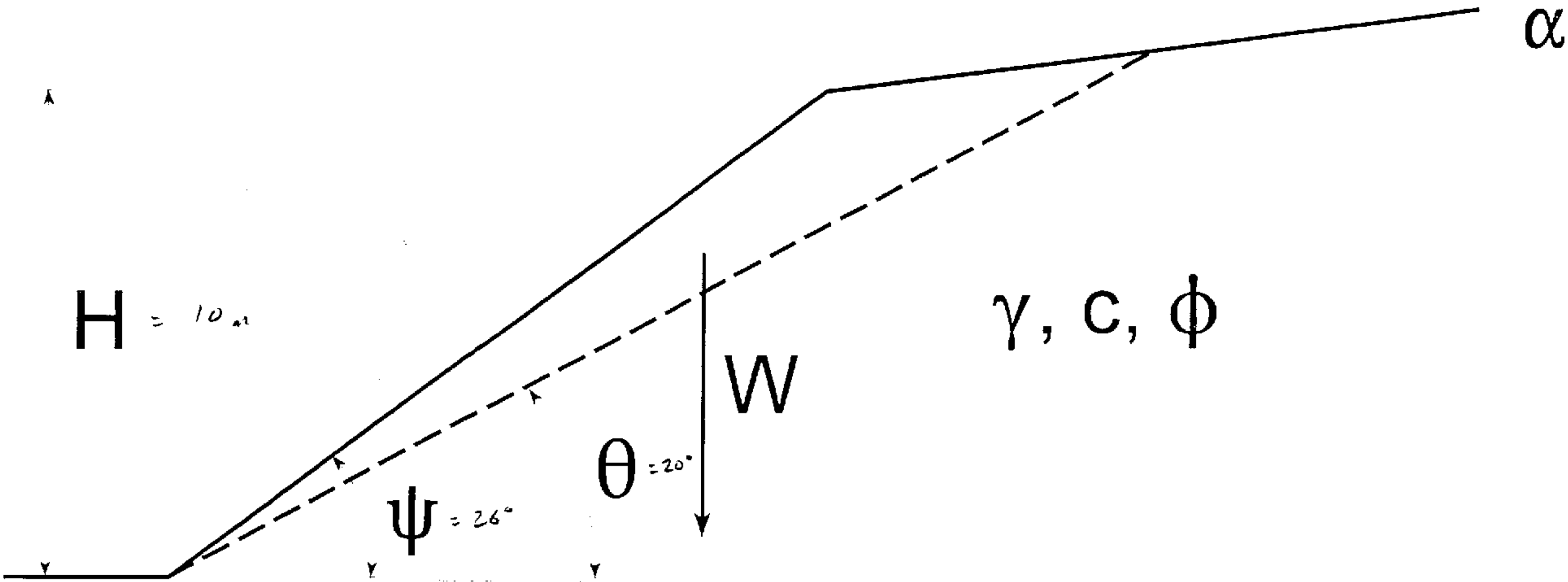

Figure 3. Geometry and notation for example Culmann problem.